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Perturbative Approach to Nonrenormalizable Theories

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Using a perturbatively nonrenormalizable and non-perturbatively finite example (delta-function-type potential in nonrelativistic quantum mechanics), we illustrate that one can develop a perturbative approach for a nonrenormalizable theory. The key idea is the introduction of an additional expansion parameter which allows us to eliminate infinities from the perturbative expressions. The generated perturbative series reproduce the expansion of the exact analytical solution.

1. INTRODUCTION

In the present paper we suggest a perturbative approach to nonrenormalizable theories. In this work renormalizability is understood in the framework of perturbation theory: a theory is called renormalizable when divergences are canceled in each order of the expansion in a coupling constant.

We assume that a given nonrenormalizable theory is finite and therefore the singularities appearing in a series generated by perturbation theory are fictitious. Fictitious singularities which cancel in expressions for physical quantities have been analyzed earlier by simple examples of a Wilson loop (Japaridze and Turashvili, 1989) and of the fermion mass (Japaridze *et al.*, 1991). We suggest a heuristic approach allowing us to avoid these singularities and which is the minimal extension of the renormalization procedure.

The latter states that renormalization is nothing else but the expression of physical quantities in terms of physical quantities (Collins, 1984; Weinberg, 1995). Specifically, when the Lagrangian contains N parameters, one calculates N physical quantities σ_i, $i = 1, 2, \ldots, N$, resolves iteratively N parame-

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ters in terms of σ, and substitutes these relations into the expression for σ_{N+1}. In renormalizable theories the regularization in the expression for the σ_{N+1} can be removed, i.e., the resulting series contain no divergences.

The proposed scheme is based on the introduction of an additional expansion parameter. So the difference with the above procedure is that in the case of N input parameters we consider $N + 1$ (and not N) regularized expressions for physical quantities.

The question which immediately arises is how the physical quantities can be defined in a nonrenormalizable theory. The problem is highly nontrivial. Even quantum electrodynamics contains surprises, e.g., it turns out that the fermion–fermion elastic scattering amplitude is nonphysical and only the inclusive cross sections can be made finite (Weinberg, 1995). Besides the well-established symmetry arguments (a physical quantity is presented by an operator, commuting with constraints; Dirac, 1964) the rest of the criteria are on an intuitive footing. We assume that *S*-matrix elements calculated in a finite and nonrenormalizable theory are finite (generally speaking, this statement may be not valid even in renormalizable theories; Japaridze and Turashvili, 1998).

The idea that successful introduction of additional expansion parameter can solve problems of infinities in perturbation theory can be illustrated by the following example: suppose "physical quantities" of our "theory" are given by the following three functions:

$$f_1 = x, \qquad f_2 = \frac{1}{x}, \qquad f_3 = \sin x + \cos \frac{1}{x}$$

If f_1 is chosen as an expansion parameter, it is impossible to express all remaining quantities as a series in positive powers of f_1 ("perturbation theory" leads to a series with infinite coefficients). The "theory" is "nonrenormalizable." The same is true when f_2 or f_3 is chosen as an expansion parameter. But if one introduces besides f_1 the additional expansion parameter f_2, the "physical quantity" f_3 can be expanded in terms of f_1 and f_2:

$$f_3 = \sin f_1 + \cos f_2 = 1 + f_1 + \frac{1}{2} f_2^2 + \ldots$$

Note that f_3 can be expressed as a function of f_1 and f_2 in an infinite number of different ways because these two parameters are not independent, but only a particular choice of functional dependence on these two parameters allows us to expand f_3 in positive powers of f_1 and f_2. Although these parameters are not independent, one can extract their numerical values from "experiment" and afterward calculate f_3 perturbatively.

At the moment we have no proof that the resulting series will reproduce correctly the features of the exact solution. The simplest way to estimate the efficiency of the method is to compare the series with the exact solution. We are not aware of realistic, exactly solvable nonperturbatively finite and perturbatively nonrenormalizable field-theoretic models. Therefore we will consider the exactly solvable quantum mechanical problem of a delta-function-type potential. Since ultraviolet divergences are considered as a trace of short-distance singularities, this potential, describing a contact interaction, is relevant in a discussion of the problems of divergences in field theory. Some examples of regularization and renormalization of delta-function potentials in nonrelativistic quantum mechanics have been considered (Beg and Furlong, 1985; Jackiw, 1991; Gosdzinsky and Tarrach, 1991; Weinberg, 1991; Manuel and Tarrach, 1994; Kaplan *et al.*, 1996).

In the present paper we apply our approach to perturbatively nonrenormalizable and non-perturbatively finite quantum mechanical models.

We show that the resulting series reproduce the expansions of exact analytic expressions.

2. FINITE AND NONRENORMALIZABLE QUANTUM MECHANICAL MODEL

We consider the example given by the following potential:

$$\langle x|V|x'\rangle = [C + C_2(\nabla^2 + \nabla'^2)]\delta(x - x')\delta(x) \tag{1}$$

with two (yet) unspecified parameters C and C_2. One could object that the problem is not mathematically well defined. Note that we do not seek much physics in this potential. For our illustrative purposes we take as a definition of the model the cutoff regularized potential with the subsequent removal of cutoff. For a definition of contact interaction in quantum mechanics see, e.g., Gosdzinsky and Tarrach (1991), Kirsch (1992), and Manuel and Tarrach (1994). So we impose a cutoff in the momentum space and henceforth work with the regularized potential:

$$V_R(p', p) = [C + C_2(p'^2 + p^2)]\theta(\Lambda - p')\theta(\Lambda - p) \tag{2}$$

The exact amplitude, expressed in terms of two renormalized parameters, is finite (Phillips *et al.*, 1998). In other words it contains only two arbitrary parameters which are fixed from two requirements on physical quantities. The model is perturbatively nonrenormalizable, i.e., to remove divergences in the framework of perturbation theory one has to include an infinite number of additional counter terms into the potential. Thus the standard perturbative renormalization technique leads to the conclusion that the physical quantities depend on an infinite number of arbitrary parameters.

Below we omit lengthy calculations and quote the results only.

Simple power counting leads to the following expansion of the T-matrix ($T \equiv S - 1$) in coupling constants C and C_2:

$$T(E) = \sum_{n_1,n_2=0}^{\infty} C^{n_1} C_2^{n_2} \Lambda^{n_1+3n_2-1} B_{n_1,n_2}(E, \Lambda) \tag{3}$$

where the coefficients $B_{n_1,n_2}(E, \Lambda)$ contain only nonpositive powers of Λ.

As a starting point of perturbative analysis, we solve iteratively the Lippman–Schwinger equation for $T(E)$ and obtain

$$T(E) = \sum_{\beta,\beta_2,i,i_3,i_5=0}^{\infty} \Omega_{\beta\beta_2 i i_3 i_5}(E) C^{\beta} C_2^{\beta_2} I^{i} I_3^{i_3} I_5^{i_5} \tag{4}$$

with finite coefficients $\Omega_{\beta\beta_2 i i_3 i_5}(E)$. The integrals I, I_3, and I_5 (see Appendix) diverge as a linear, third, and fifth powers of the cutoff regulator. So far, the amplitude requires renormalization.

We drop inverse powers of regulator Λ in $I(E)$ and write

$$I(E) = -\frac{1}{\pi^2} \Lambda - \frac{i}{2\pi} (2E)^{1/2} = I_1 - \frac{i}{2\pi} (2E)^{1/2} \equiv I_1 + W(E) \tag{5}$$

Nonperturbative renormalization can be carried out by choosing the scattering length a and the effective range r_e as the renormalization parameters and thus fixing C and C_2 (Phillips *et al.*, 1998).

Designating $C_R = 2\pi a = T|_{E=0}$, $C_{2R} = r_e C_R^2/8\pi$, we express C and C_2 iteratively as a power series of C_R and C_{2R}:

$$C_2 = C_{2R} - 2C_R C_{2R} I_1 + 3C_R^2 C_{2R} I_1^2 - 4C_R^3 C_{2R} I_1^3 - \frac{3}{2} C_{2R}^2 I_3 + \ldots \tag{6}$$

and

$$C = C_R - C_R^2 I_1 + C_R^3 I_1^2 - C_R^4 I_1^3 - 2C_R C_{2R} I_3 + C_R^5 I_1^4 + 6C_R^2 C_{2R} I_1 I_3 - C_R^6 I_1^5 - 12C_R^3 C_{2R} I_1^2 I_3 - C_{2R}^2 I_5 + \ldots \tag{7}$$

The substitution of (6) and (7) into (4) leads to the following expression for the amplitude:

$$T(E) = T_R(C_R, C_{2R}, E) + 16C_{2R}^2 E^2 I_1 + \ldots \tag{8}$$

where we wrote explicitly the first term, remaining divergent after renormalizing iteratively C and C_2. For the explicit expression of the finite part T_R see the Appendix; $W(E)$ is defined in (5).

In this simple potential model the T-matrix is a sum of bubble diagrams. All the divergences are contained in these bubbles (one-loop subdiagrams). Simple power counting shows that each loop can contain divergences $\sim\Lambda^5$,

Λ^3, and Λ [it is obvious from the expression for a loop, which reads $\int d^3k(C + C_2(p^2 + k^2))(C + C_2(p'^2 + k^2))/k^2$]. The maximum power Λ^5 originates from the term C_2^2 when both vertices of the bubble are proportional to C_2. The coefficient of this divergence does not depend on momentum and this divergence is included in the renormalization of the coupling constant C. The coefficient of the next divergence, $\sim\Lambda^3$, contains momenta square and is absorbed by a counter term for C_2, i.e., it is included in the renormalization of coupling constant C_2. So, renormalization of C and C_2 allows us to eliminate divergences Λ^5 and Λ^3 from the perturbative expression for the T-matrix.

For the last type of divergence, $\sim\Lambda$, the term $16C_{2R}^2E^2I_1$ remains after the renormalization of the parameters of the potential, and thus the maximum divergence of the n-loop contribution is Λ^n. The standard renormalization technique leads to the conclusion that the theory is nonrenormalizable and the physical quantities depend on an infinite number of arbitrary parameters. In our case, when the exact solution is available we know that this conclusion is wrong—the model is self-consistent in the sense that all the divergences cancel.

It is convenient to reparametrize the renormalized coupling C_{2R} as $C_{2R} = \frac{1}{2}yC_R^2$ and write a formula analogous to (3):

$$\begin{aligned} T(E) &= \sum_{n_1,n_2=0}^{\infty} C_R^{n_1}C_{2R}^{n_2}\Lambda^{n_1+n_2-1}B_{n_1,n_2}^R(E, \Lambda) \\ &= \sum_{n_1,n_2=1}^{\infty} y^{n_2}C_R^{n_1+2n_2}\Lambda^{n_1+n_2-1}B_{n_1,n_2}^R(E, \Lambda) \\ &= \sum_{n_1,n_2=1}^{\infty} y^{n_2}(C_R\Lambda)^{n_1+n_2-1}C_R^{n_2+1}B_{n_1,n_2}^R(E, \Lambda) \end{aligned} \tag{9}$$

where $B_{n_1,n_2}^R(E, \Lambda)$ contains only nonpositive powers of Λ. The terms with $n_2 = 0$ do not contain divergences because the potential with $C_2 = 0$ is renormalizable. The terms with $n_2 = 1$ also do not contain divergences because these divergences are absorbed into the renormalization of C_2. Substituting $C_{2R} = \frac{1}{2}yC_R^2$ into (8), we obtain

$$T(E) = T_R(C_R, y, E) + 4C_R^4y^2E^2I_1 + \ldots \tag{10}$$

Now we are in a position to demonstrate how the method works. According to our assumption (and as is known from the exact solution), $T(E)$ is finite. Consequently, the sum of terms containing positive powers of Λ should be finite itself. We introduce quantity α related to this sum by the following relation:

$$\alpha(\mu) = \frac{1}{4y^2C_R^3\mu^2}\,[T(\mu) - T(\mu)|_{\Lambda=0}] = C_R I_1 + \ldots \qquad (11)$$

To ensure that $\alpha(\mu)$ is not complex, we take μ negative.

From (9)–(11) it follows that α contains only positive powers of $C_R I_1$, y, and C_R. Thus $C_R I_1$ can be solved iteratively from (11) as a series in α, y, and C_R.

Expressing $C_R I_1$ in terms of α and substituting into (10) leads to

$$T(E) = T_R\,(C_R, y, E) + 4C_R^3 y^2 E^2\alpha + \ldots \qquad (12)$$

In general, the finite part is also affected by α, but in a given order we have the same T_R. (12) is an expansion of amplitude in terms of y, C_R, and α.

To compare the series (12), originating from the perturbation theory, with the exact solution, we have to express latter in terms of the same α. For the renormalized exact solution we have

$$T(E) = \frac{C_R(1 + C_R I_1 + 2EC_R y)}{1 + C_R I_1 - 2EI_1 y C_R^2 - W(E)C_R(1 + C_R I_1 + 2EyC_R)} \qquad (13)$$

and in the $\Lambda \to \infty$ limit

$$T(E) = \frac{C_R}{1 - 2EyC_R - W(E)C_R} \qquad (14)$$

Evidently, the solution is finite in the limit $\Lambda \to \infty$.

We introduce the quantity

$$\alpha(\mu) = \frac{1}{4\mu^2 y C_R^3}\,[T(\mu^2) - T(\mu)|_{I_1=0}]$$

$$= \frac{C_R I_1}{[1 - C_R W(\mu)(1 + 2\mu C_R)][1 - C_R W(\mu)(1 + 2\mu C_R) + C_R I_1(1 - 2\mu C_R - C_R W(\mu))]} \qquad (15)$$

Now, extracting $C_R I_1$ from (15) and substituting into (13), we obtain an expression for the exact solution:

$$T(E) = \frac{N}{D} \qquad (16)$$

where N and D are given in the Appendix. In the limit $\Lambda \to \infty$ the expression for T is the same, but now α is

$$\alpha(\mu) = \frac{1}{[1 - C_R W(\mu) - 2\mu y C_R^2 W(\mu)][1 - 2\mu y C_R - W(\mu)C_R]} \qquad (17)$$

The substitution of α from (15) into $T = N/D$ results in expression (13) for $T(E)$, while substitution of α from (17) leads to (14).

So the exact solution, expressed in terms of y, C_R, and α, is given by expression (16), and introducing an additional expansion parameter in the renormalized Lippman–Schwinger series (8), we obtain the series (12). It is straightforward to check that the perturbative series (12) is the expansion of the exact result given by (16). A tedious calculation shows that the same statement is true in the next order.

Summarizing, although the quantum mechanical problem with contact interaction (1) is perturbatively nonrenormalizable, by introducing an additional expansion parameter α we have obtained a finite perturbative expression for the amplitude. It turns out that this series reproduces the expansion of the exact solution (14). If the model under consideration were realistic, one could extract the value of α from experiment and compare our result to data, while the standard renormalization technique requires the introduction of an infinite number of counter terms and leads to the conclusion that the theory has no predictive power.

3. DISCUSSION

Introducing an additional expansion parameter, we were able to reproduce correctly the exact result for a perturbatively nonrenormalizable and non-perturbatively finite quantum mechanical problem. As is clear from the discussion of the toy model with f_1, f_2, f_3 the approach is successful for finite theories which face singularities in perturbative expansion. Roughly speaking, introduction of the additional expansion parameter is equivalent to regularizing the remaining singularities in terms of the expression for the extra physical quantity. Therefore, it is not surprising that this procedure leads to finite series. Considering speculations on the nonperturbative finiteness of quantum gravity (Nakanishi and Ojima, 1990; Isham, 1995; Thiemann, 1998), one could try to apply the scheme for the gravitational interaction. Calculation again results in a finite series (Gegelia *et al.*, 1995). Since in the case of quantum gravity exact solutions as well as experimental data are not available, there is nothing to be compared with the resulting series. Therefore, the result of Gegelia *et al.* (1995) can be considered for now as a mathematical exercise only.

Although the suggested procedure leads to perturbative series with finite coefficients, we have no criterion for distinguishing actually infinite theories (when the exact solutions diverge or do not exist) from the non-perturbatively finite ones. The same is true for any procedure when the character of the convergence is not known. The problem of convergence of perturbation theory series, still unsolved completely for renormalizable theories, can be traced back to Dyson (1951).

From our point of view, the suggested scheme, leading to finite series for nonrenormalizable theories, again confirms that renormalizability is not the fundamental requirement for selecting a correct theory (Weinberg, 1995).

APPENDIX

The integrals appearing in (4) are

$$I(E) = 2 \int \frac{d^3k}{(2\pi)^3} \frac{\theta(\Lambda - k)}{2E^+ - k^2}$$

$$= -\frac{1}{\pi^2}\left[\Lambda + \frac{(2E)^{1/2}}{2} \ln \frac{1 - (2E)^{1/2}/\Lambda}{1 + (2E)^{1/2}/\Lambda}\right] - \frac{i}{2\pi}(2E)^{1/2}$$

$$I_n = -2 \int \frac{d^3k}{(2\pi)^3} k^{n-3}$$

where $E^+ \equiv E + i0$ and the finite part T_R is

$$T_R = C_R + C_R^2 W(E) + C_R^3 W(E)^2 + C_R^4 W(E)^3 + C_R^5 W(E)^4 + C_R^6 W(E)^5$$
$$+ 8EW(E)C_R C_{2R} + 4EC_{2R} + 12EW(E)^2 C_R^2 C_{2R}$$
$$+ 16\, EW(E)^3 C_R^3 C_{2R} + 16E^2 W(E) C_{2R}^2$$

The numerator N and denominator D of the expression (16) are equal to

$$N = C_R + 2EyC_R^2 - \alpha(\mu)[2yC_R^2(E - \mu) - 2yC_R^3 W(\mu)(E - \mu) - 4E\mu y^2 C_R^3]$$
$$\times [1 - W(\mu)C_R - 2\mu W(\mu) y C_R^2]$$

and

$$D = 1 - C_R W(E) - 2EW(E)yC_R^2 + \alpha(\mu)[1 - C_R W(\mu) - 2\mu W(\mu) y C_R^2]$$
$$\times [2yC_R(\mu - E) - 2W(\mu)(\mu - E)yC_R^2 + 2W(\mu)W(E)(\mu - E)yC_R^3]$$
$$+ \alpha(\mu)[1 - C_R W(\mu) - 2\mu W(\mu) y C_R^2][4\mu E(W(\mu)$$
$$- W(E))y^2 C_R^3 + 2W(E)yC_R^2]$$

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REFERENCES

Beg, M. A., and Furlong, R. C. (1985). *Physical Review D*, **31**, 370.
Birse, M. C., McGovern, J. A., and Richardson, G. (1998). hep-ph/9807302.
Collins, J. C. (1984). *Renormalization*, Cambridge University Press, Cambridge.
Dirac, P. A. M. (1964). *Lectures on Quantum Mechanics*, Yeshiva University Press, New York.
Dyson, F. J. (1951). *Physical Review*, **85**, 631.
Gegelia, J., Japaridze, G., Kiknadze, K., and Turashvili K. (1995). hep-th/9507037.
Gosdzinsky, P., and Tarrach, R. (1991). *American Journal of Physics*, **59**, 70.
Isham, C. J. (1995). gr-qc/9510063.
Jackiw, R. (1991). In *M. A. Beg Memorial Volume*, World Scientific, Singapore.
Japaridze, G., and Turashvili, K. (1989). *Zeitschrift für Phyzik C*, **44**, 651.
Japaridze, G., and Turashvili, K. (1998). *International Journal of Theoretical Physics*, **37**, 865.
Japaridze, G., Khelashvili, A., and Turashvili, K. (1991). In *Proceedings of XII Conference Quarks-90*, World Scientific, Singapore.
Kaplan, D.., Savage, M. B., and Wise, M. B. (1996). *Nuclear Physics B*, **478**, 629.
Kirsch, W. (1992). In *Ideas and Methods in Quantum and Statistical Physics*, Cambridge University Press, Cambridge.
Manuel, C., and Tarrach, R. (1994). *Physics Letters B*, **328**, 113.
Nakanishi, N., and Ojima, I. (1990). *Covariant Operator Formalism of Gauge Theories and Quantum Gravity*, World Scientific, Singapore.
Phillips, D. B., Beane, S. R., and Cohen, T. D. (1998). *Annals of Physics*, **263**, 255.
Thiemann, T. (1998). *Classical and Quantum Gravity*, **15**, 1282; gr-qc/9705019.
Weinberg, S. (1991). *Nuclear Physics B*, **363**, 3.
Weinberg, S. (1995). *The Quantum Theory of Fields*, Cambridge University Press, Cambridge.